Partial hedging and equity-linked life insurance

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Introduction: Theoretical Motivations

Using hedging for pricing and risk management is common in financial mathematics. It is usual to construct a strategy that exactly replicates the cash flows of a contingent claim. When the exact replication of a contingent claim (c.c.) is not possible, a financial strategy with a cash flow that is "close enough" to the payoff of the contingent claim in some probabilistic sense can be considered as a (partial, imperfect) hedging strategy.

Introduction: Theoretical Motivations

Mean-variance, *quantile*, and *efficient hedging* are well-known types of partial hedging. This talk is devoted to quantile hedging where one aims to find a hedge that maximizes the probability of successful hedging under a constraint on the initial capital. Applying quantile hedging to equity-linked life insurance contracts allows one to obtain equations separating financial and insurance risks embedded in the contracts and propose a methodology for effective risk-management.

Introduction: Theoretical Motivations

References on equity-linked life insurance include:

- Brennan and Schwartz (1976, 1979)
- Boyle and Schwartz (1977)
- Bacinello and Ortu (1993)
- Aase and Person (1994)
- Ekern and Persson (1996) : fair price valuation
- Moeller (1998, 2001) : mean-variance hedging

The above-mentioned authors studied contracts with a fixed or deterministic guarantee at maturity.

Introduction: Practical Motivations

According to the Annuity Fact Book (NAVA, 2004)

1. Total industry Annuity Net Assets grew in the period 1994-2003 from 307.5 to 993.9 billions of dollars;
2. Total industry Annuity Sales grew in the period 1994-2003 from 50.2 to 126.4 billions of dollars.

Introduction: Problems

- Appropriate pricing and hedging are needed for such contracts to *reduce default* for the seller as well as to *avoid mispricing* and potentially damaging regulatory decisions.
- Develop flexible and practical risk management programs that take into account *survival behavior* among the groups of potential clients.
- Introduce and study adequate *mortality models* (risk assessment measures) to guarantee optimal results for modern actuarial analysis.

Introduction: In this paper we

- consider a financial market driven by diffusion and jump-diffusion processes;
- study equity-linked life insurance contracts with a stochastic guarantee;
- use an imperfect hedging technique (and a new technique based on risk measures);
- develop mortality modeling that takes into account new trends in mortality;
- illustrate our results using real data.

Description of Models

We consider a financial market consisting of a non-risky asset $B_t = e^{r\,t}$, $t \geq 0$, $r \geq 0$ is a continuously compounded rate of return, and risky asset S_t (or assets S_t^1, S_t^2) on a probability space $\big(\Omega,\, \mathcal{F},\, P\big)$. The filtration $\mathcal{F}$ consists of the flow of information on the evolution of prices of risky asset(s).

Model 1

Suppose that prices of the risky asset follow the Black-Scholes model

$$dS_t = S_t \left(\mu \, dt + \sigma \, dW_t\right) ,$$

where W is a Wiener process, μ and σ are the rate of return and the volatility of the asset S, respectively, and S_0 is the initial investment. This market is complete and has the unique risk-neutral probability defined by

$$Z_T = \exp\left\{ -\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T \right\} .$$

Model 2

We now suppose that risky assets follow the two dimensional Black-Scholes model

$$dS_t^i = S_t^i \left(\mu_i \, dt + \sigma_i \, dW_t^i \right), \quad i = 1, 2,$$

where W_t^i are two Wiener processes with

$$cov(W_t^1, W_t^2) = \rho \, t, \quad |\rho| \leq 1 \, .$$

We assume that $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$ since S^2 is supposed to provide a flexible guarantee and should be less risky than S^1.

Model 3

Here we suppose that risky assets follow the jump-diffusion model

$$dS_t^i = S_{t-}^i \left(\mu_i \, dt + \sigma_i \, dW_t - \nu_i \, d\Pi_t\right), \quad i = 1, 2,$$

where Π is a Poisson process with a positive intensity λ, and ν_i describe the sizes of the jumps of S^i. We also assume $\mu_1 > \mu_2$, $\sigma_1 > \sigma_2$, $\nu_i < 1$ and

$$\frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\sigma_2 \nu_1 - \sigma_1 \nu_2} > 0 \, .$$

Model 3

Then the unique risk-neutral probability has the following density with respect to the initial probability

$$Z_T = \exp\left\{\alpha^* W_T - \frac{(\alpha^*)^2}{2} T + (\lambda - \lambda^*)t + \ln \frac{\lambda^*}{\lambda} \Pi_t\right\},$$

where (α^*, λ^*) is the unique solution of the system

$$\mu_i - r = -\sigma_i \alpha^* + \nu_i \lambda^*, \quad i = 1, 2.$$

Parameter $\lambda^* > 0$ can be interpreted as a new intensity of the Poisson process with respect to the risk-neutral probability.

Strategies and Contingent Claims

Let $\pi_t = (\beta_t, \gamma_t^1, \gamma_t^2)$ be a portfolio that consists of β_t units of the non-risky asset and γ_t^1, γ_t^2 units of risky assets. Its value is given by

$$X_t^\pi = \beta_t B_t + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2.$$

Every $\mathcal{F}_T$-measurable nonnegative random variable H is called a *contingent claim*. A self-financing strategy π is a perfect hedge for H if $X_T^\pi \geq H$ (a.s.). According to the option pricing theory of Black-Scholes-Merton, such strategy does exist and is unique for a given contingent claim. Its initial value is $X_0^\pi = e^{-rT}\,\mathbf{E}^* H$.

Insurance Setting

Let random variable $T(x)$ on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ represent the remaining lifetime of a person of age x. Let

$$_{T}p_{x} := \widetilde{P}\{T(x) > T\}$$

be a survival probability for the next T years of the insured. It is reasonable to assume that $T(x)$ does not depend on the evolution of the financial market and, therefore we can assume that $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ are independent.

Insurance Setting

We study contracts with fixed and flexible stochastic guarantees with a possible payoff at the maturity. Due to independency of "financial" and "insurance" parts of contracts, we can consider them on the product probability space $\left(\Omega \times \widetilde{\Omega},\, \mathcal{F} \times \widetilde{\mathcal{F}},\, P \times \widetilde{P}\right)$. The payoff of a contract with a fixed guarantee is linked to the prices described by Model 1:

$$H\big(T(x)\big) = \max\left\{S_T,\, S_0 e^{gT}\right\} I_{\{T(x)>T\}}\,,$$

where g is a guaranteed rate.

Insurance Setting

The contracts with a flexible guarantee are considered in the framework of the Models 2 and 3. In this case the payoff at the maturity is

$$H\big(T(x)\big) = \max\left\{S_T^1,\, S_T^2\right\} I_{\{T(x)>T\}}\,.$$

These contracts are often referred to as ***equity-linked life insurance contracts*** with fixed and flexible guarantees. Our further discussion is devoted to contacts with flexible guarantees within the Model 3.

Brennan-Schwartz Price

The price of such contract is given by (Brennan and Schwartz (1976))

$$ {}_TU_x = \mathbf{E}^* \times \widetilde{\mathbf{E}}\, H(T(x)) = {}_T\, p_x\, S_0 + {}_T\, p_x \mathbf{E}^* \left(S_T^1 - S_T^2\right) e^{-rT}. $$

We note that this price is strictly less than the "fair" price of H:

$$ {}_TU_x < \mathbf{E}^* H\, e^{-rT} =: H_0\,, $$

thus the perfect hedging of H is not possible and one has to look for alternative types of hedging.

Quantile Hedging

We assume that the self-financing portfolio π has a budget constraint

$$X_0^\pi \le X_0 < H_0 \,.$$

We say that the portfolio π^* is a ***quantile hedge*** if

$$P\big\{A(X_0^{\pi^*}, \pi^*)\big\} = \max_{\pi : X_0^\pi \le X_0} P\big\{A(X_0^\pi, \pi)\big\} \,,$$

where

$$A(X_0^\pi, \pi) := \big\{\omega : \; X_T^\pi \ge H\big\}$$

is a *successful hedging set*.

$\exists$! **Result** (Foellmer and Leukert (1999)): Assume that $A^* \in \mathcal{F}_T$ is a solution to the problem

$$P(A^*) = \max_{A \in \mathcal{F}_T : \mathbf{E}^*(H\, I_A) \le X_0} P(A)\,,$$

then there is a unique quantile hedge π^* which is a perfect hedge for a modified claim $H_{A^*} = H\, I_A$. The structure of the ***maximal successful hedging set*** is $A^* = \left\{ Z_T^{-1} > aH \right\}$, where constant a is defined by X_0.

Quantile Hedging: equity-linked life insurance

Due to the structure of the Brennan-Schwartz price we now solve the quantile hedging problem for the following ***embedded option***

$$\left(S_T^1 - S_T^2\right)^+ \cdot I_{\{T(x)>T\}} \,.$$

From the general $\exists!$ result we obtain the key relation

$${}_T p_x = \frac{\mathbf{E}^*\left(S_T^1 - S_T^2\right)^+ \cdot I_{A^*}}{\mathbf{E}^*\left(S_T^1 - S_T^2\right)^+}$$

that connects the financial and insurance parts of the contract.

Quantile Hedging: equity-linked life insurance

We can write the maximal successful hedging set in the form

$$A^* = \{Z_T^{-1} > a(S_T^1 - S_T^2)^+\} = \left\{ \left(\frac{S_T^1}{S_T^2}\right)^{-q} \geq c\left(\frac{S_T^1}{S_T^2} - 1\right)^+ \right\},$$

where constants c and q depend on the parameters of the model. Properties of the characteristic equation

$$y^{-q} = c\,(y-1)^+$$

allow us to obtain the following result.

Quantile Hedging: Main Result

Suppose that the jump-diffusion Model 3 satisfies the condition

$$\frac{(\mu_1 - r)\nu_2 - (\mu_2 - r)\nu_1}{\sigma_2\nu_1 - \sigma_1\nu_2} = \frac{\sigma_1\,\sigma_2}{\sigma_2 + (\sigma_1 - \sigma_2)^2}\,.$$

Then the characteristic equation has a unique solution d_n on the set $\{\Pi_T = n\}$ and it is possible to balance the survival probability of an insured and the financial risk associated with the contract:

Quantile Hedging: Main Result

$$
{}_Tp_x = 1 - \frac{\sum_{n=0}^{\infty} p_n^* \Big(\vartheta_n^1 \Phi\big(b_+(\vartheta_n^1, d_n\vartheta_n^2, T)\big) - \vartheta_n^2 \Phi\big(b_-(\vartheta_n^1, d_n\vartheta_n^2, T)\big)\Big)}{\sum_{n=0}^{\infty} p_n^* \Big(\vartheta_n^1 \Phi\big(b_+(\vartheta_n^1, \vartheta_n^2, T)\big) - \vartheta_n^2 \Phi\big(b_-(\vartheta_n^1, \vartheta_n^2, T)\big)\Big)} \, ,
$$

where $\vartheta_n^i = (1-\nu_i)^n \, e^{\nu_i \lambda^* T}$ and $p_n^* = \exp\big\{-\lambda^* T\big\} \frac{(\lambda^* T)^n}{n!}$. Constants d_n can be found from the condition

$$
P\left(\frac{S_T^1}{S_T^2} \le d_n \Big| \Pi_T = n \right) = 1 - \varepsilon \, ,
$$

Quantile Hedging: Main Result

i.e. by fixing the probability of failure to hedge the embedded option on the set $\{\Pi_T = n\}$. We obtain

$$d_n = \left(\frac{1-\nu_1}{1-\nu_2}\right)^n \exp\left\{(\sigma_1 - \sigma_2)\,\Phi^{-1}(1-\varepsilon)\,\sqrt{T} + \left((\mu_1 - \mu_2) - \frac{\sigma_1^2 - \sigma_2^2}{2}\right) T\right\}.$$

The proof is based on the conditional log-normality of prices and on the representation

$$W_T = \big(1 + (\sigma_2 - \sigma_1)\big) W_T - \big(\sigma_2 - \sigma_1\big) W_T\,.$$

Diversification of Mortality Risk

We now pool together all homogeneous clients of the same age, life expectancy and the same investment preferences, and consider cumulative claims $l_{x+T} \cdot H$, where l_{x+T} is the number of insureds survived to time T from the group of size l_x.
Let $H_{0,\varepsilon}$ be the initial capital of a quantile hedge π_ε with the risk level ε, then

$$P\Big(X_T^{\pi} \geq H\Big) = 1 - \varepsilon\,.$$

Diversification of Mortality Risk

Suppose that

$$\widetilde{P}\big(n_\alpha \geq l_{x+T}\big) = 1 - \alpha\,,$$

where $\alpha \in (0,1)$ characterizes the level of a cumulative mortality risk. Then, due to the independence of insurance and financial risks, we have

$$\begin{aligned} P \times \widetilde{P}\,\big(l_x\, X_T^\pi \geq l_{x+T}\, H\big) &\geq P\Big(X_T^\pi \geq \frac{n_\alpha}{l_x} H\Big)\, \widetilde{P}\big(n_\alpha \geq l_{x+T}\big) \\ &\geq (1-\varepsilon)(1-\alpha) \geq 1 - (\varepsilon + \alpha)\,. \end{aligned}$$

Diversification of Mortality Risk

This allows to introduce a reduced price of an individual contract

$$H_{\varepsilon,\alpha} = \frac{n_\alpha}{l_x} \, {}_T p_x \, \mathbf{E}^*(H \, e^{-rT}) \, ,$$

that is associated with the cumulative risk $\varepsilon + \alpha$.

Risk Management Procedure

Step 1. Fix ε, the level of financial risk or the probability that the insurance company will fail to hedge successfully the claim H with a maturity guarantee.

Step 2. Find a survival probability ${}_Tp_x$ of a client whose mortality risk would offset the financial risk. The obtained survival probability will give a quantile price of the contract ${}_Tp_x\,\mathbf{E}^*(H\,e^{-rT})$.

Step 3. Fix α, the level of insurance risk or the probability that the number of clients alive at maturity will be greater than expected.

Risk Management Procedure

Step 4. Using a binomial distribution (or another appropriate frequency distribution) with parameters l_x and ${}_T p_x$, calculate n_α, the number of clients that would expose the insurance company to a cumulative mortality risk α.

Step 5. Calculate the price of the contract $\frac{n_\alpha}{l_x}\, {}_T p_x \, \mathbf{E}^*(H\, e^{-rT})$ that reflects both financial and insurance risks.

Step 6. Repeat the same steps for all possible combinations of ε and α. Design a grid for financial and insurance risks reflecting the corresponding values of equity-linked insurance contracts as a function of the values of risks.

Numerical Analysis: Model 3

We modify the approach of Mancini (2004) to obtain estimates for the parameters. The financial indices Russell 2000 and S&P 500 are considered as risky assets. It is supposed that the first index is more risky than the second as it consists of small stocks. Therefore, we consider the S&P 500 as a flexible guarantee. We work with monthly observations over 25 years from January 1979 to December 2004 to get estimates of the parameters of Model 3:

Numerical Analysis: Model 3

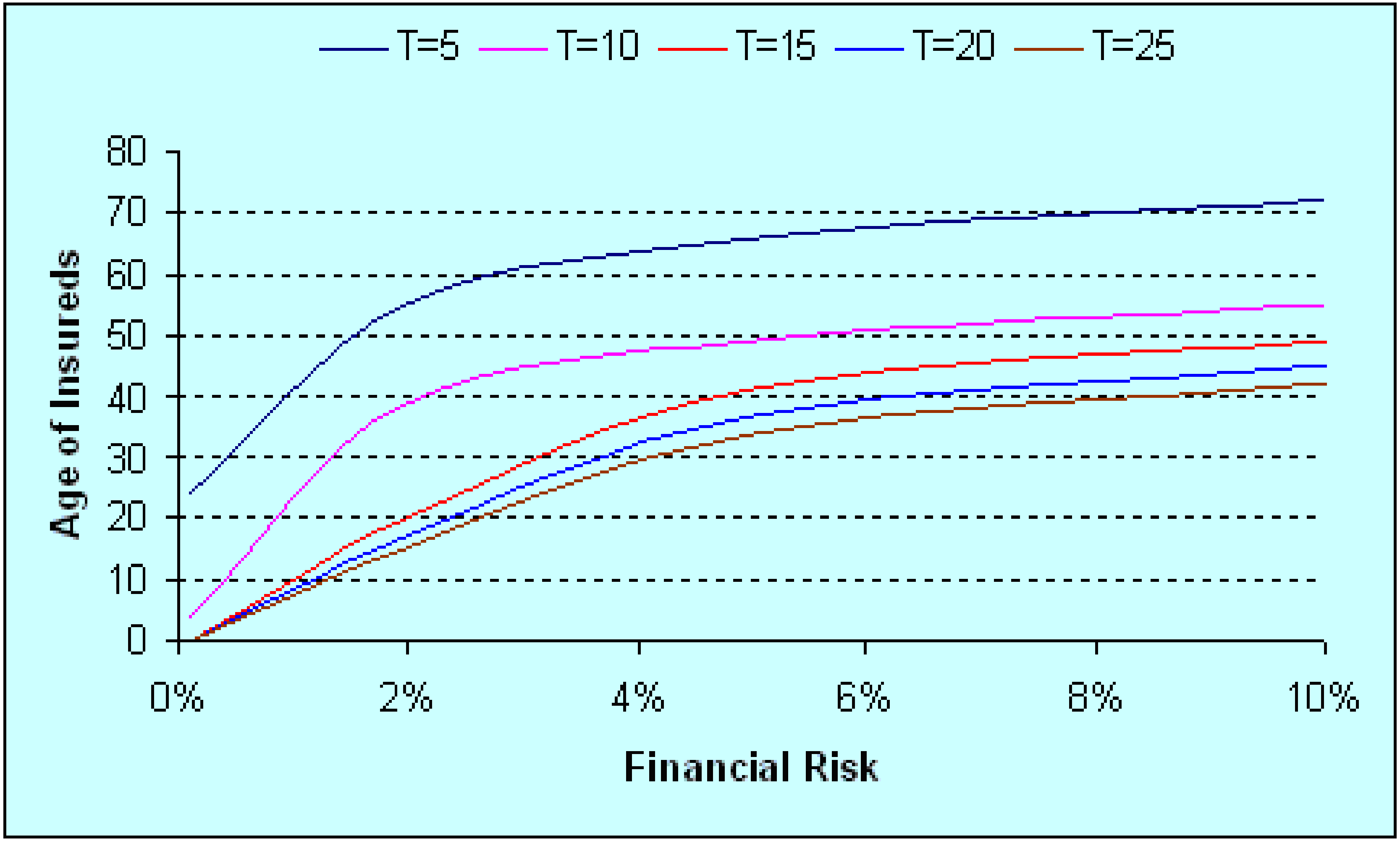
$$
\begin{aligned}
\lambda &= 0.07 \\
\mu_1 &= 0.11 \quad \sigma_1 = 0.2 \quad \nu_1 = 0.04 \\
\mu_2 &= 0.1 \quad \sigma_2 = 0.16 \quad \nu_2 = 0.04
\end{aligned}
$$

We consider an initial investment $S_0 = 100$. Let terms of contracts be $T = 5, 10, 15, 20, 25$ years and the risk-free rate be $r = 6\%$.

Numerical Analysis: Model 3

We now calculate survival probabilities of the clients for different levels of financial risk ε. The corresponding ages of the clientele are taken from the ***mortality table*** UP94@2015 which gives the best estimates for mortality projected to year 2015.

Age of Insured



Acceptable Financial Risk

Age	T = 5	T = 10	T = 15	T = 20	T = 25
30	0.14%	1.2%	3.0%	3.3%	3.9%
35	0.16%	1.5%	3.4%	4.3%	5.6%
40	0.22%	2.3%	4.6%	6.3%	8.7%
45	0.30%	3.1%	6.9%	9.8%	13.6%
50	0.50%	5.5%	11.0%	15.3%	19.8%
55	1.00%	10%	17.4%	22.0%	28.5%

Prices of contracts with $\alpha = 2.5\%$

Age	T = 5	T = 10	T = 15	T = 20	T = 25
30	4.18	5.89	7.18	8.23	9.12
35	4.18	5.88	7.15	8.18	8.89
40	4.17	5.86	7.11	8.06	8.53
45	4.17	5.83	7.02	7.66	7.71
50	4.15	5.77	6.68	6.94	6.59
55	4.12	5.56	6.14	5.95	5.00
Margrabe	4.23	5.98	7.32	8.45	9.45

Further Developments: Mortality Modeling

Instead of Mortality Tables one can use theoretical Mortality Models:

Gompertz $\mu_x = B\,c^x\,, \quad c > 1,\ B > 0$;

Makeham $\mu_x = A + B\,c^x\,, \quad c > 1,\ B > 0,\ A > -B$;

Lee-Carter $\ln(m_{x,t}) = a_x + b_x\,k_t + \psi_{x,t}$,

where μ_x is the force of mortality, $m_{x,t}$ is the death rate for age x and year t; k_t is the mortality index modeled as a random walk.

Further Developments: Mortality Modeling

We now consider Model 1 and a contract with the payoff $H = \max\left\{S_T, S_0\, e^{gT}\right\}$. Using quantile hedging we calculate

$$ {}_T p_x = \frac{\mathbf{E}^*(H\, I_{A^*})}{\mathbf{E}^*(H)} \quad \text{where} \quad A^* = \left\{Z_T^{-1} > a^* H\, e^{-rT}\right\}. $$

We further compute

$$ {}_T p_x = 1 - \frac{e^{(g-r)T}\,\Phi(M^*)}{\Phi(\sigma\sqrt{T} - N^*) + e^{(g-r)T}\,\Phi(N^*)} $$

and the initial price of such contract

$$ {}_{T}U_x = S_0\,\Phi(\sigma\sqrt{T} - N^*) + S_0\,e^{(g-r)T}\Big[\Phi(N^*) - \Phi(M^*)\Big]\,, $$

where

$$ N^* = \frac{g - r + \frac{\sigma^2}{2}}{\sigma}\sqrt{T}\,, \qquad M^* = \frac{\ln(a^* S_0) + \left(g - r + \frac{\theta^2}{2}\right)T}{\theta\sqrt{T}} $$

and

$$ \theta = \frac{\mu - r}{\sigma}\,. $$

Two Perspectives of Risk Management

1. choose a client (i.e. choose ${}_Tp_x$) and find the maximal probability of successful hedging;
2. fix the default (i.e. choose ε), then calculate ${}_Tp_x$ and therefore choose an appropriate client for the given contract with the specified level of risk.

Numerical Results

Financial Data TSX/S&P index daily prices 1995-2005

$$\mu = 9.11\% \ \sigma = 15.73\% \ S_0 = 9246.7 \ r = 5.61\% \ g = 7\%$$

Mortality Data USA, Sweden, Japan annual death rates 1959-1999 (HMD)

RM Perspective 1

The first approach of quantile hedging: the firm maximizes the probability of successful hedging given a limited initial capital. Suppose a 60-year old client approaches the firm with the intention of buying a contract that will allow him/her to receive the maximum of TSX/S&P fund value, currently at CAD 9246.70, and a guaranteed amount (based on the guaranteed rate of 7 percent) in 3 (10, 20) years, respectively. The client must be alive to collect the payoff.

RM Perspective 1

Probabilities of successful hedging (in percent) based on Gompertz (G), Makeham (M), Lee-Carter (LC) models

T	G_{US}	M_{US}	LC_{US}	G_S	M_S	LC_S	G_J	M_J	LC_J
3	98.2	98.2	98.4	98.7	98.7	98.9	98.6	98.5	98.6
10	94.1	94.1	95.4	95.5	95.5	96.4	95.1	95.0	95.5
20	81.5	81.6	87.7	83.8	83.7	87.6	82.2	82.2	83.0

RM Perspective 1: Summary

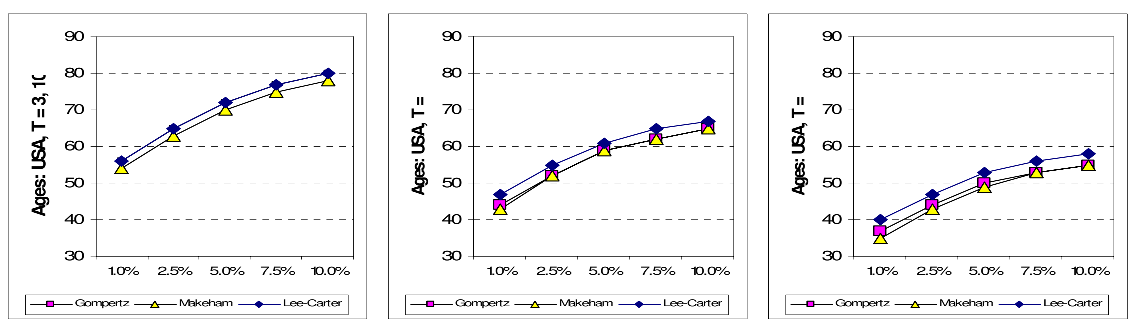
- Hedge less successful with longer maturities
- Gompertz and Makeham are almost identical
- Bigger differences for US, some for Sweden
- Lee-Carter predicts higher hedging success
- Lee-Carter are more accurate in mortality modeling
- financing problems for insurance firms
- undervalued retirement costs in US (maybe Sweden)

RM Perspective 2

Suppose an insurance firm sells contracts, but requires that the probability of default risk, ε, does not exceed some specified value. Based on this chosen risk profile, the company calculates the minimal amount of funds needed to hedge the payoff with the prescribed probability and, via one of the three mortality models, determines the ages of clients who would pay a fair price for the contract based on different default levels.

RM Perspective 2

The results are given in a graphical format in the following Figure.



Pricing with other Risk Measures

We note that is natural to take into account not only the possibility of losses but also the expected size of losses. Therefore, another risk measure called CVaR (Conditional Value at Risk) can supplement our analysis.

$$VaR_{1-\varepsilon}(\pi) = \inf \Big\{ z : \; P\{H - X_T^{\pi}(x) \leq z\} \geq 1 - \varepsilon \Big\} \, ,$$

$$CVaR_{1-\varepsilon}(\pi) = \mathbf{E}\Big(H - X_T^{\pi}(x) \, \Big| \, H - X_T^{\pi}(x) \geq VaR_{1-\varepsilon}(\pi) \Big) \, .$$