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Global Portfolio Optimization: A Least Discrimination Alternative to Black-Litterman

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Outline

- **Black-Litterman (BL) Model Revisited**
  - BL assumptions, results and critique
  - BLS: the BL singular case
  - BLC: the BL conditional case
  - Illustration
- **Optimal Pay-offs**
  - Maximum expected utility (EU) pay-off
  - Active pay-off
  - Generalized relative entropy distance (GRE)
- **The Least Discriminatory (LD) Model**
  - The Least Discrimination Principle
  - LD distributions with views on expected returns and covariances

Black-Litterman Revisited

- "Quantitative asset allocation models have not played the important role they should" (Black &Litterman [1992]). Why? They often led to **badly behaved** portfolios (**unrealistic** and **unstable**)

- Practitioners have coped with these problems by using:
  - **Robust estimators** (Jorion [1986]) ⇒ priors
  - **Constraints** (Jaggannathan [2003]) ⇒'corner' solutions
  - **Robust optimizer** (Fabozzi [2007]) ⇒ conservative MaxMin

and/or the

BL seek to improve probabilistic forecasts

BL Approach

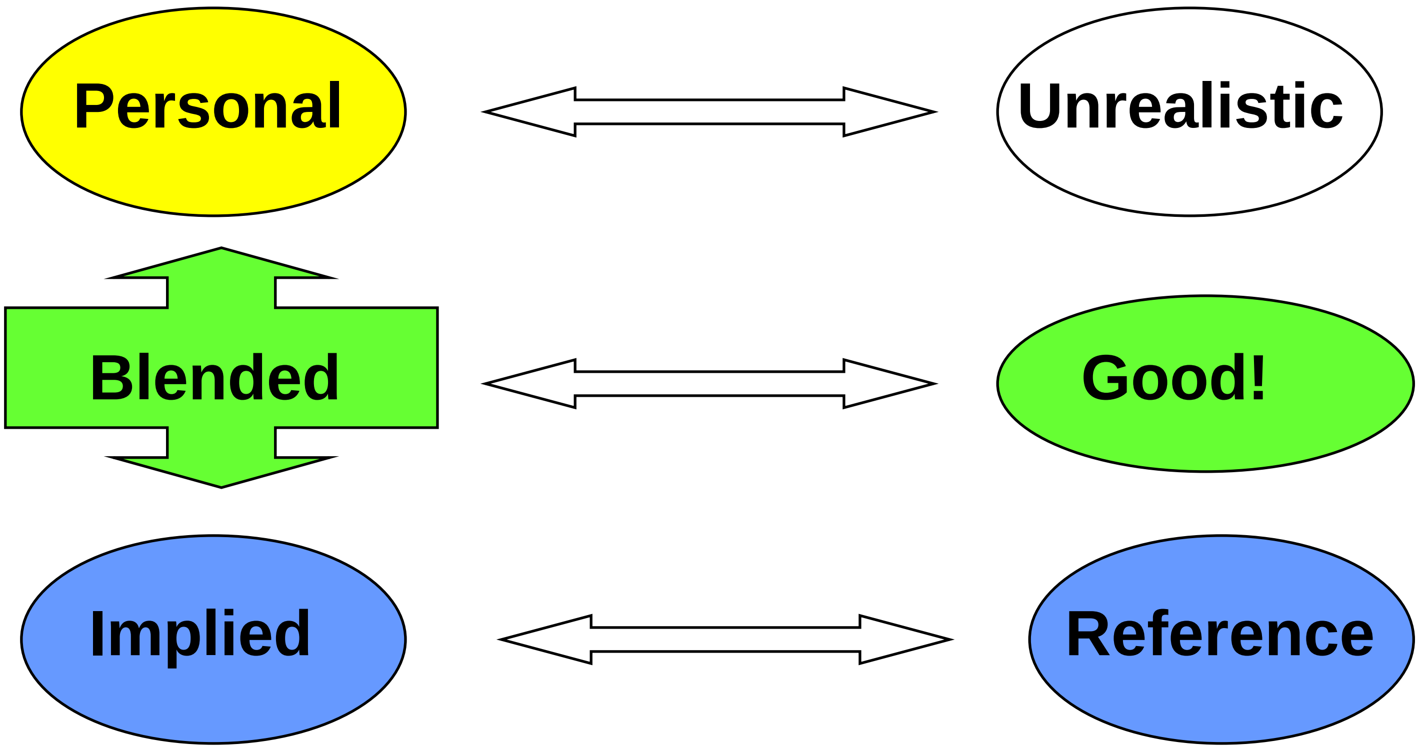
1. Choose a **reference portfolio**
2. Derive a **market implied forecast**
3. **Blend** personal views with market implied forecast

Blending can produce optimal portfolios as closed as desired to the reference portfolio

BL blending technique assumes **uncertainties in** personal and market implied views of **expected returns!!!**

BL Approach

Forecast ⇐ M-V Analysis ⇒ Optimal Portfolio



The Market Implied Forecast

Markowitz M-V $\Rightarrow$ $$\boldsymbol{\pi} = \gamma \boldsymbol{\Sigma\omega} \quad (1)$$

- $\boldsymbol{\omega}$ vector of n risky asset allocations in reference portfolio
- Σ covariance matrix; sample estimate from historical data, perhaps adjusted for volatilities implied by option prices
- γ investor's coefficient of risk aversion
- $\boldsymbol{\pi}$ vector of **expected excess returns implied** by γ, $\boldsymbol{\Sigma}$, and $\boldsymbol{\omega}$ assuming the **reference portfolio is optimal**.

BL assume that implied expected returns μ are uncertain with mean given by (1) and covariance matrix $\tau\boldsymbol{\Sigma}$, $\tau < 1$

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma}) \quad (2)$$

Personal Views

- Personal views limited to expected returns on a few assets
- Views about expected returns $\boldsymbol{\mu}$ are expressed by a set of k linear equations

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{q} + \boldsymbol{\varepsilon} \qquad (3)$$

- $\boldsymbol{\varepsilon}$ represents the uncertainty in the views $\mathbf{q}$. The components of $\boldsymbol{\varepsilon}$ are assumed to be independently normally distributed, so that

$$\mathbf{P}\boldsymbol{\mu} \sim N(\mathbf{q}, \boldsymbol{\Omega}) \qquad (4)$$

$\boldsymbol{\Omega}$ is a diagonal matrix

BL Blending Method

- Combine respective pdfs as if from two **independent sources of information**
- Posterior distribution $\mu \sim N(\mathbf{p}, \mathbf{M})$ has an expected value and a covariance matrix that depend on the assumed uncertainties in each of the two information sources:

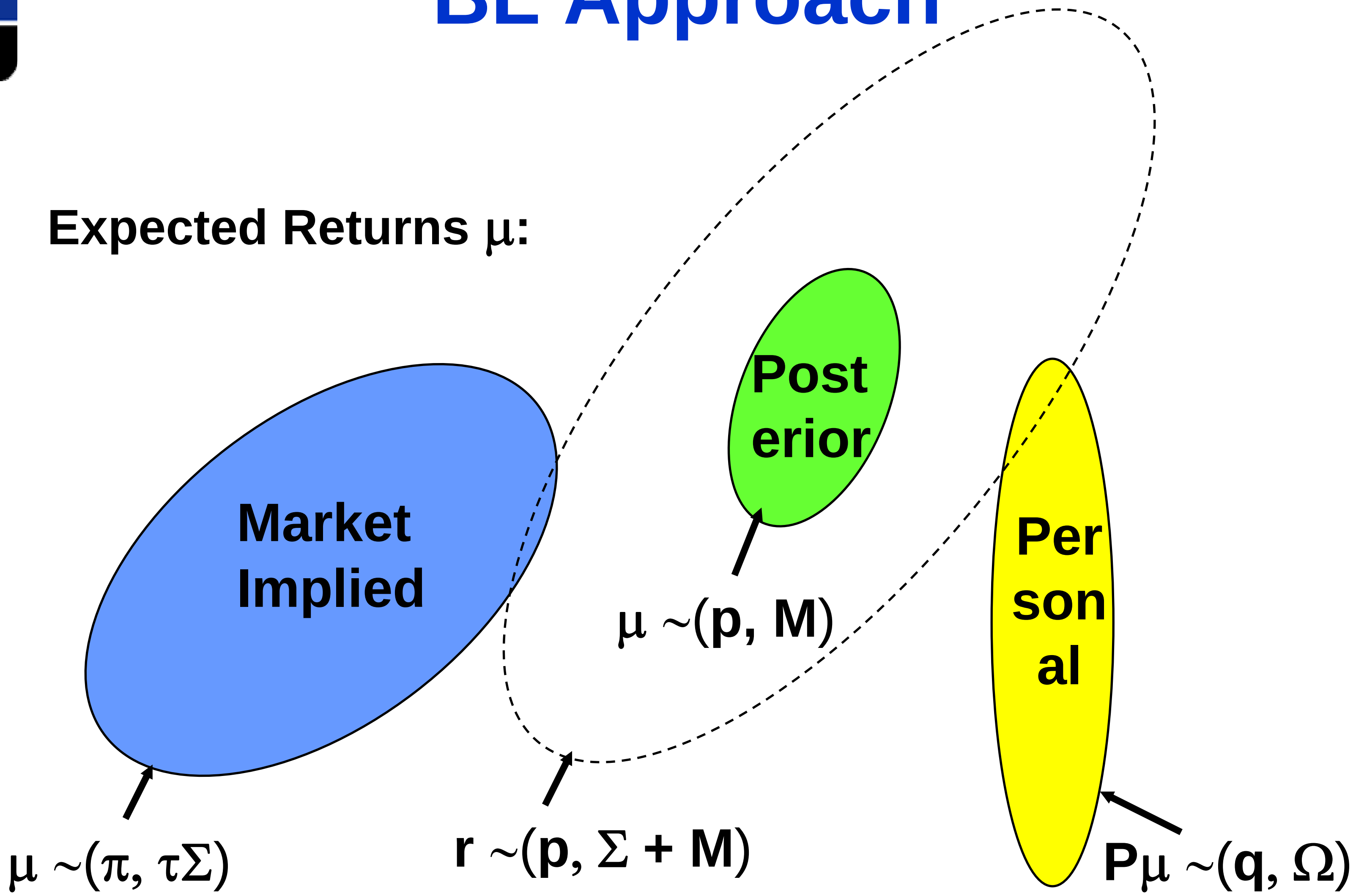
$$\mathbf{p} = [\Sigma^{-1} + \mathbf{P}'\tau\Omega^{-1}\mathbf{P}]^{-1} [\Sigma^{-1}\pi + \mathbf{P}'\tau\Omega^{-1}\mathbf{q}] \qquad (5)$$

$$\mathbf{M} = \tau[\Sigma^{-1} + \mathbf{P}'\tau\Omega^{-1}\mathbf{P}]^{-1} \qquad (6)$$

- Distribution of asset returns:

$$\mathbf{r} \sim N(\mathbf{p}, \Sigma + \mathbf{M}) \qquad (7)$$

BL Approach


Expected Returns μ:
Market Implied
Post erior
Per son al
μ ~(p, M)
μ ~(π, τΣ)
r ~(p, Σ + M)
Pμ ~(q, Ω)

Critique of BL

- Uncertainties about expected values?
- Increase of uncertainty about returns
- Ad hoc choice of uncertainty parameters $\tau\Sigma$ and Ω
- Independence between market and personal forecasts
- Personal views limited to expected returns
- Two steps in one:
  1. **adjusting** forecast of correlated assets
  2. **blending** with market implied forecast

→ are the results **credible?** Can they be **justified** more logically? Can they be **extended** to other views?

The Singular Case: BLS

Uncertainties in expected returns is only a BL artifice to calculate posterior expected returns

Note that $\mathbf{p}$ and $\mathbf{M}$ in (5) and (6) depend on $\tau\boldsymbol{\Omega}^{-1}$ but not on τ and $\boldsymbol{\Omega}$ separately

So, shrink τ and $\boldsymbol{\Omega}$ to zero keeping $\tau\boldsymbol{\Omega}^{-1}$ constant: $\mathbf{p}$ remains constant and $\mathbf{M}$ vanishes – the posterior distribution of expected returns becomes **singular**

$\tau\boldsymbol{\Omega}^{-1}$ can be defined in terms of a diagonal matrix of **credibility weights C for personal views**:

$$\tau\boldsymbol{\Omega}^{-1} = \text{Diag}[(\mathbf{P\Sigma P'})^{-1}]\mathbf{C}(\mathbf{I} - \mathbf{C})^{-1} \qquad (8)$$

The conditional case: BLC

When some personal views are held with certainty ($\mathbf{C} = 1$ or $\tau\mathbf{\Omega}^{-1}$ infinite), the distribution of expected returns becomes the distribution conditional on the views. The posterior expected returns are given by:

$$\mathbf{p} = \boldsymbol{\pi} + \boldsymbol{\Sigma}\mathbf{P}'[\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}']^{-1}[\mathbf{q} - \mathbf{P}\boldsymbol{\pi}] \qquad (9)$$

This result can be obtained directly by minimizing the quadratic distance between $\mathbf{p}$ and $\boldsymbol{\pi}$

$$d = (\mathbf{p} - \boldsymbol{\pi})'\boldsymbol{\Sigma}^{-1}(\mathbf{p} - \boldsymbol{\pi})$$

subject to the views on expected returns:

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{q}$$

Illustration of BL, BLS & BLC

MARKET IMPLIED FORECAST

Two risky assets A, B: $\sigma_A = 10\%$, $\sigma_B = 20\%$, $\rho_{AB} = +62.5\%$ (eg. A = G8 equity index, B = BRIC equity index) and a risk-free asset C

Reference portfolio (M): $\omega_A = \omega_B = 0.5$, $\omega_C = 0$

Risk aversion: $\gamma = 4$

Under BLS

(1) $\Rightarrow$ market implied **$\pi_A = 4.5\%$, $\pi_B = 10.5\%$**

Under BL, with $\tau = 0.09$: $\sigma_A = 10.44\%$, $\sigma_B = 20.88\%$

(1) $\Rightarrow$ market implied **$\pi_A = 4.95\%$, $\pi_B = 11.45\%$**

Illustration of BLS & BLC

A PERSONAL VIEW

Expected return **$\mu_A = 5.5\%$**

Might also be expressed as:

P1: $\mu_A = 5.5\%$ and $\mu_B = 10.5\%$

P2: $\mu_A = 5.5\%$, no view on μ_B

P3: $\mu_B - \mu_A = 5\%$

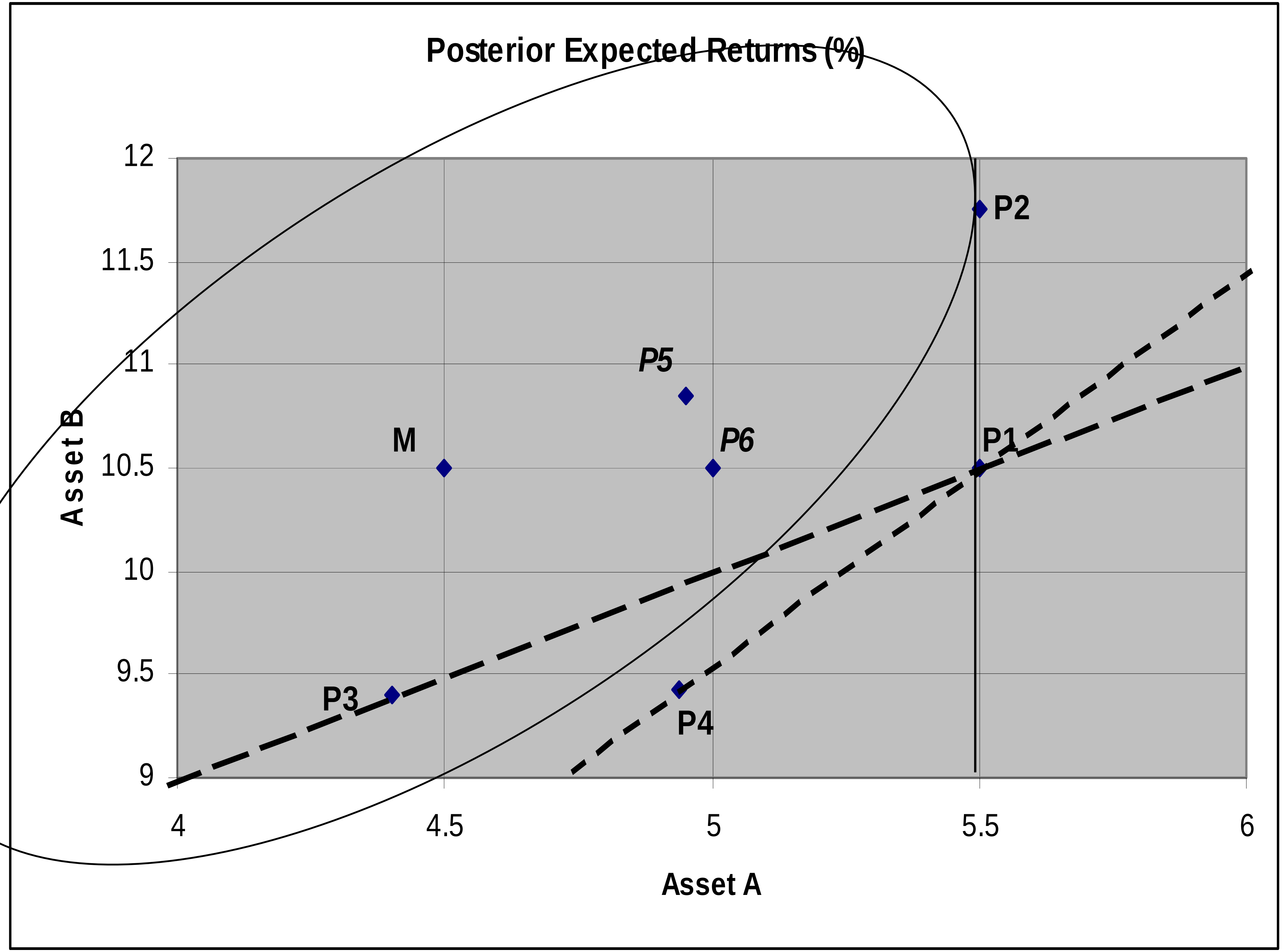
P4: $11\mu_B – 21\mu_A = 0\%$

Question: What are the corresponding posterior expected returns and optimal allocations?

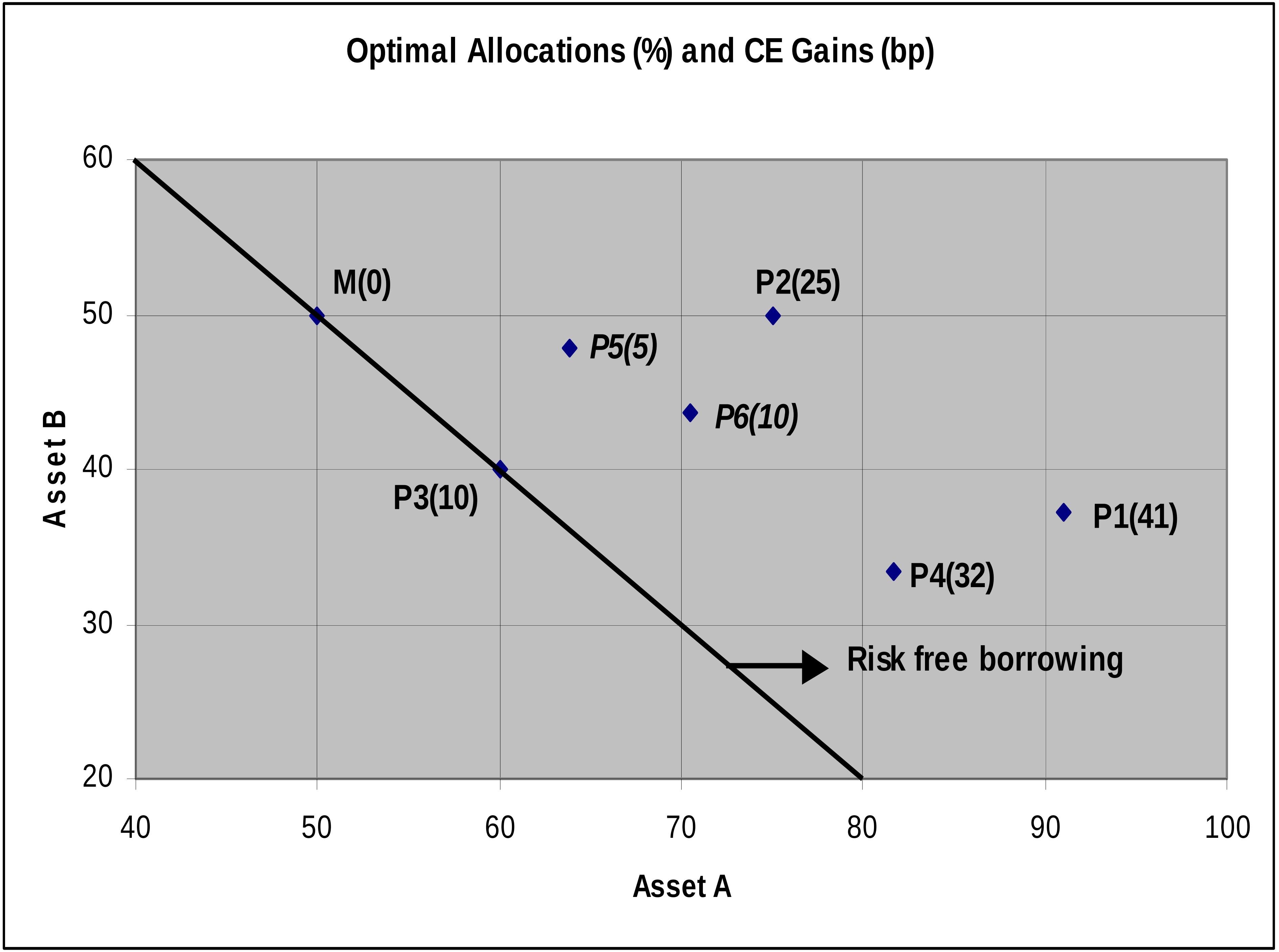
Posterior Returns and Optimal Portfolios under BLS & BLC

Market (M) and Personal Views (P) in %		Posterior Exp.		Optimal Allocations (%)		
		A	B	A	B	C
M	$\mu_A = 4.5, \mu_B = 10.5$	4.5	10.5	50	50	0
P1	$\mu_A = 5.5, \mu_B = 10.5$	5.5	10.5	91	37.2	-28.2
P2	$\mu_A = 5.5$	5.5	11.75	75	50	-25
P3	$\mu_B - \mu_A = 5.0$	4.4	9.4	60	40	0
P4	$11\mu_B - 21\mu_A = 0$	4.94	9.43	81.7	33.4	-15.1
P5	50% P1 and 50% M	4.95	10.85	63.8	47.8	-11.7
P6	Interpol. (M, P1)	5	10.5	70.5	43.6	-14.1

Results are very sensitive to ways views are expressed



Posterior Expected Returns (%)
12
11.5
11
10.5
10
9.5
9
4
4.5
5
5.5
6
Asset A
Asset B
P2
P5
M
P6
P1
P3
P4



Optimal Allocations (%) and CE Gains (bp)
Asset B
Asset A
60
50
40
30
20
40
50
60
70
80
90
100
M(0)
P2(25)
P5(5)
P6(10)
P3(10)
P1(41)
P4(32)
Risk free borrowing

Optimal Pay-offs

Programme:

1. Seek optimal (maximum expected utility) pay-off $f(\mathbf{r} \mid q, p)$ given personal forecast $p(\mathbf{r})$ and risk neutral forecast $q(\mathbf{r})$. **Non-linear pay-offs are generalizations of Markowitz optimal portfolios**
2. Evaluate the Certainty Equivalent (CE) of the optimal pay-off $\Rightarrow$ relative entropy distance
3. Evaluate the gain in CE from adding an optimal active pay-off to a reference portfolio $\Rightarrow$ generalized relative entropy distance (GRE)
4. Illustrate with forecasts expressed as multivariate normal distributions (MVN)

Optimal (Maximum EU) Pay-off

Seek pay-off $f(\mathbf{r})$ that maximises the investor's expected utility of wealth at maturity T:

$$\underset{f(.)}{\text{argmax}}\ E_p[u(f(\mathbf{r}) - E_q[f(\mathbf{r})] + w_0)] \qquad (10)$$

$p(\mathbf{r})$ personal forecast (subjective probability)

$q(\mathbf{r})$ risk neutral forecast (determines the cost $E_q[f(\mathbf{r})]$ of pay-off $f(\mathbf{r})$)

w_0 investor's initial wealth

$u(.)$ investor's utility function

$E[.]$ expectation operator

Opt. Pay-Off with Exponential u(.)

With $u(x) = -\exp(-\gamma x)$, γ = coefficient of risk aversion:

$$f(\mathbf{r}\,|\,q, p) = [\ln(p(\mathbf{r})/(q(\mathbf{r})) + D(q, p)]/\gamma \qquad (11)$$

where $$D(q, p) = E_q[\ln(q(\mathbf{r})/p(\mathbf{r})] \qquad (12)$$

so that $E_q\,[f(\mathbf{r}\,|\,q, p)\,] = 0$, i.e., pay-off purchased at par

Pay-off expected utility is $E_p[u(f(\mathbf{r}\,|\,q, p))] = -\exp(-D(q, p))$

Therefore, $D(q, p)/\gamma$ is the **certainty equivalent** of the pay-off

$D(q, p)$ is also known as the **relative entropy distance** between pdfs $q(\mathbf{r})$ and $p(\mathbf{r})$.

Optimal Active Pay-off

The reference pay-off $f(\mathbf{r} | q, m)$ is optimal for the market forecast $m(\mathbf{r})$. Pay-off $f(\mathbf{r} | q, p)$ is optimal for personal forecast $p(\mathbf{r})$. By difference the optimal active pay-off is:

$$f(\mathbf{r} | q, m, p) := f(\mathbf{r} | q, p) - f(\mathbf{r} | q, m)$$

$$= (\ln(p(\mathbf{r})/(m(\mathbf{r})) + D(q, p) - D(q, m))/\gamma \qquad (13)$$

The **gain in CE from adding the net active pay-off** is:

$$D(q, m, p)/\gamma = (\ln(E_q[p(\mathbf{r})/m(\mathbf{r})]) - E_q[\ln(p(\mathbf{r})/m(\mathbf{r}))])/\gamma \qquad (14)$$

$D(q, m, p)$ is a **generalized relative entropy distance**

Optimal Pay-off Illustration

Reference portfolio fully invested in risky asset with $\sigma = 15\%$,
Risk aversion: $\gamma = 4$

Implied expected return $\pi = \gamma\sigma^2\omega = 9\%$

Personal forecast : $p = 8\%$; $s = 20\%$

Optimal pay-off (from (11)):

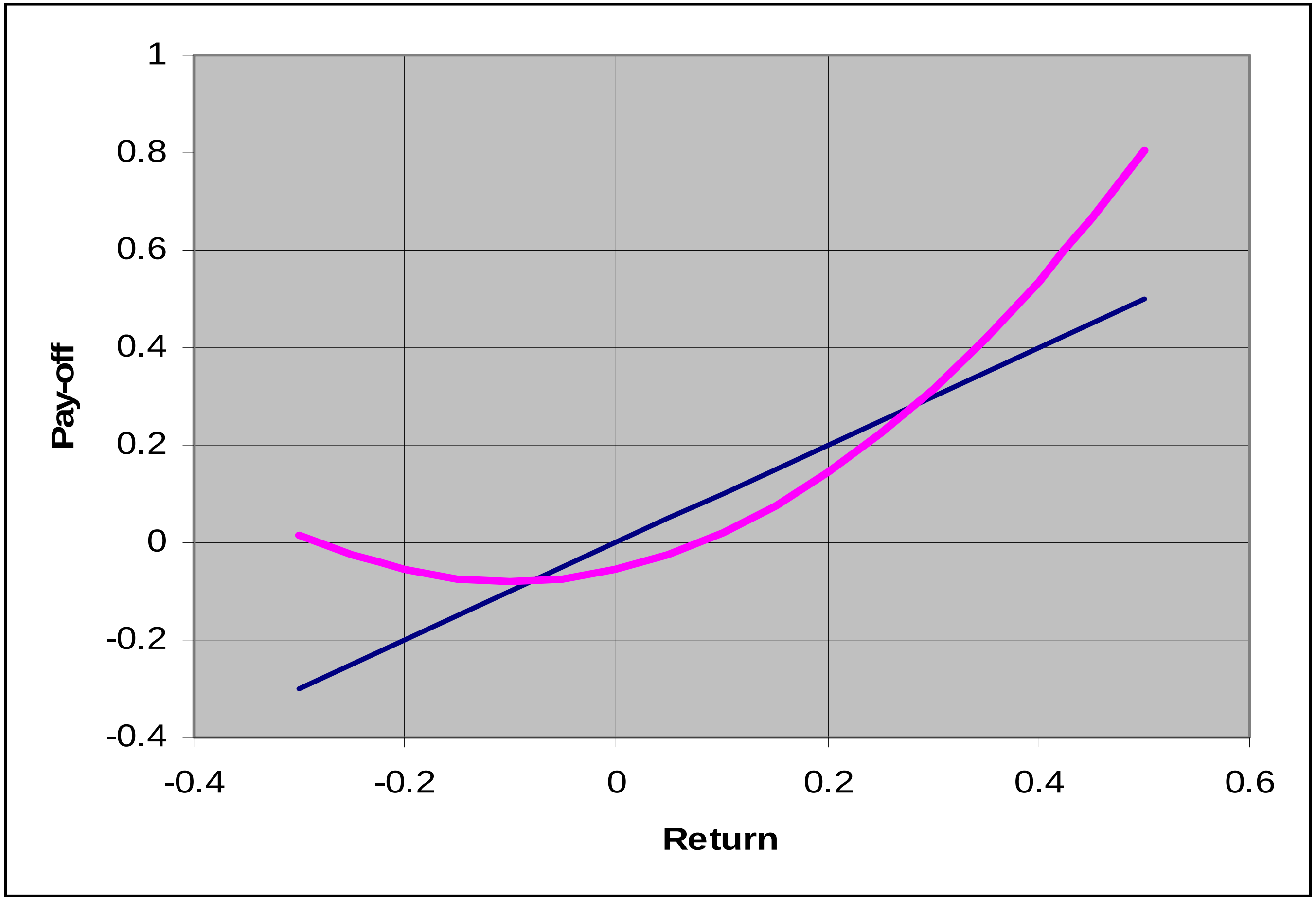
$$f(r\,|\,q, p) = (1/2\,\gamma)(r^2(\sigma^{-2} - s^{-2}) + 2rps^{-2}) = 2.43(r^2 - 0.0225) + 0.5r$$

Optimal active pay off:

$$f(r\,|\,q, m, p) = f(r\,|\,q, p) - f(r\,|\,q, m) = 2.43(r^2 - 0.0225) - 0.5r$$

The optimal pay-off is quadratic in r. It is an option like pay-off (approximate with combination of listed options or buy OTC)

Optimal Pay-off Illustration



Least Discrimination (LD) Model

LEAST DISCRIMINATION PRINCIPLE

Among all personal forecasts satisfying a set of views, select the forecast that yields the minimum increase in certainty equivalent when optimally acted upon

Rationale:

Any other forecast would lead to gains in certainty equivalent not supported by personal views and therefore not credible

Calculating LDDs from Views

Given $q(\mathbf{r})$, $m(\mathbf{r})$ and $p(\mathbf{r})$ represented by $N(0, \mathbf{\Sigma})$, $N(\boldsymbol{\pi}, \mathbf{\Sigma})$ and $N(\mathbf{p}, \mathbf{S})$, we seek parameters $(\mathbf{p}, \mathbf{S})$ for the Least Discriminatory Distribution (LDD) that

Minimise $D(q, m, p)$

subject to (views):

$$\mathbf{P}\mathbf{p} = \mathbf{q} \tag{15}$$

$$\mathbf{G}.\mathrm{Vech}(\mathbf{S}) = \mathbf{h} \tag{16}$$

Hence, minimize the Lagrangian

$$L = D(q, m, p) - \boldsymbol{\lambda}_1'(\mathbf{P}\mathbf{p} - \mathbf{q}) - \boldsymbol{\lambda}_2'(\mathbf{G}.\mathrm{Vech}(\mathbf{S}) - \mathbf{h}) \tag{17}$$

Application to Alternative Investments (AI)

Exhibit 1: Performance of the Portfolio Constituents

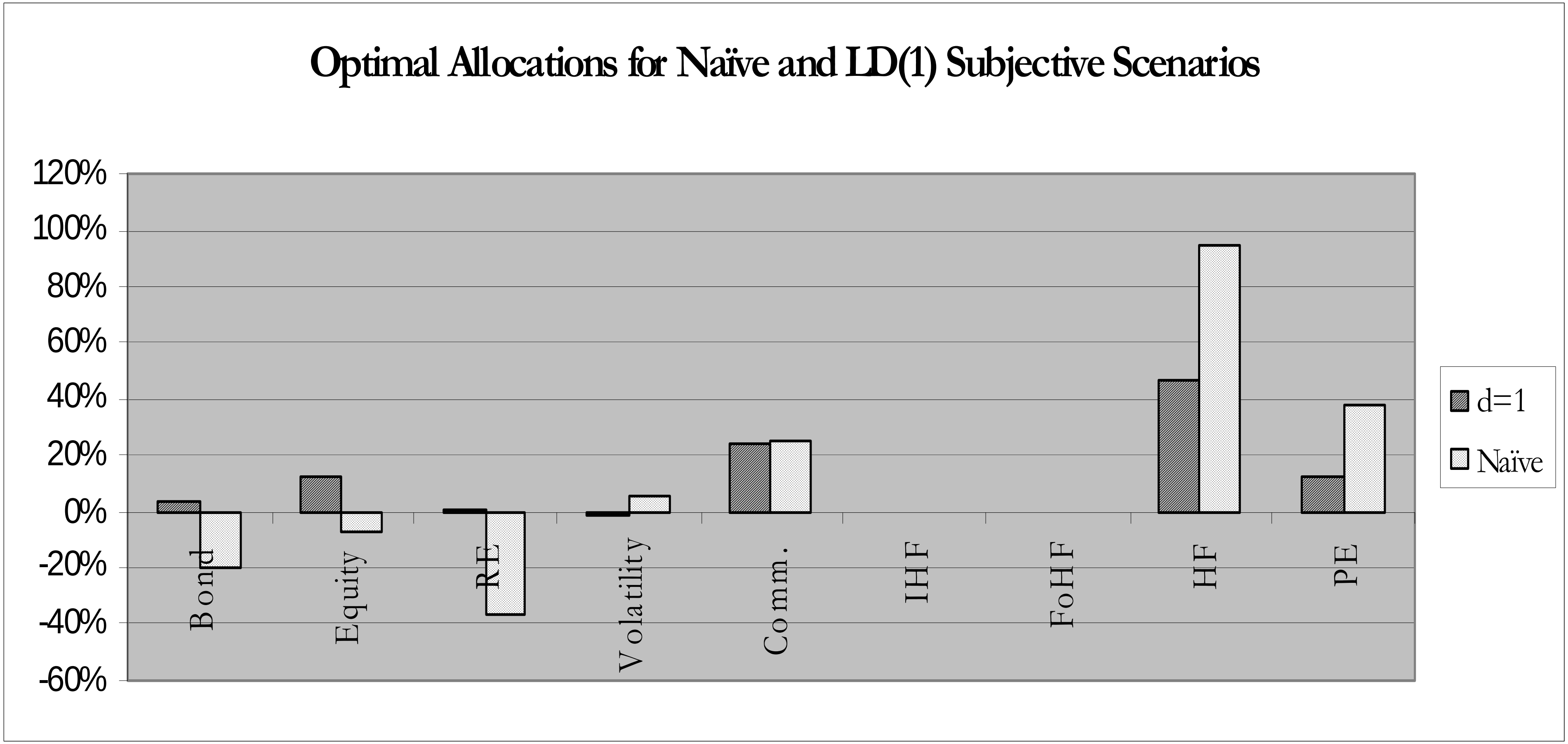
Annual Excess Returns	Bond	Equity	RE	Volatility	Comm.	IHF	FoHF	HF	PE
Mean	1.55%	13.03%	17.18%	-69.25%	5.79%	3.27%	4.80%	8.82%	22.75%
Std. Dev.	3.08%	11.09%	11.32%	37.34%	7.73%	3.71%	4.71%	6.16%	17.61%
Skewness	-0.11	0.07	-0.38	0.18	-0.15	-0.09	-0.24	-0.08	-0.06
Excess Kurtosis	0.01	0.07	0.19	0.19	0.01	-0.05	0.08	-0.04	0.02
1-mth autocorrelation	0.10	0.12	-0.28	0.08	-0.03	0.34	0.22	0.26	0.29
Sharpe Ratio	0.50	1.17	1.52	-1.85	0.75	0.88	1.02	1.43	1.29

Based on monthly arithmetic-returns for the period 1 October 2002 to 31 December 2006

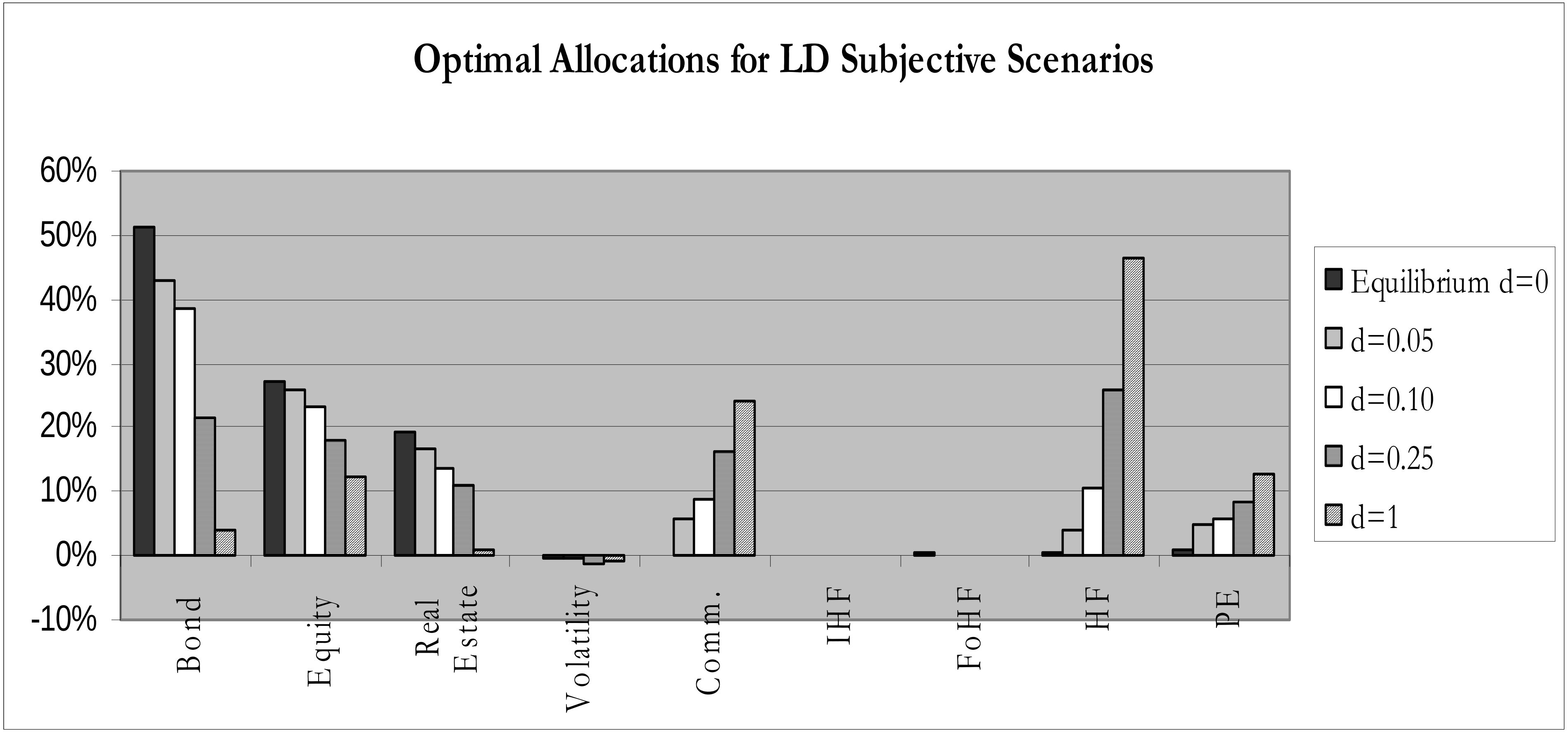
Choice of Forecasts for AI & Traditional Assets

- Choose global market portfolio as reference
- Derive implied expected returns (equilibrium returns)
- Forecast: Use as personal view the average returns observed on AI since October 2002, and for traditional assets use
  - Naïve Scenario – equilibrium returns
  - LD(1) scenario – LD adjusted returns
  - LD(d) Scenarios – blended forecasts with credibility d given to historical forecast of AI and (1 – d) to equilibrium returns

Comparison of Naïve and LD(1) Forecasts



Range of Blended LD Forecasts



Summary & Conclusions

- **BL** seek to transform a limited set of personal views into a **complete** and **reasonable probabilistic forecast**
  - But they use questionable assumptions and ad hoc parameters and their method is limited to views on expected returns
  - Use the singular limit, BLS
- The **Least Discrimination principle** offers a **more logical and general** way to achieve the same goal
  - Yields easily optimal portfolios and gains in certainty equivalent
  - With views on covariances, leads to quadratic pay-offs (options)
  - Blending market and personal forecasts still simple
- **Mode of expression of personal views remains critical**

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Appendix: GRE Between MVN Distributions

When $q(\mathbf{r})$, $m(\mathbf{r})$ and $p(\mathbf{r})$ are represented by $N(0, \boldsymbol{\Sigma})$, $N(\boldsymbol{\pi}, \boldsymbol{\Sigma})$ and $N(\mathbf{p}, \mathbf{S})$, respectively, then

$$f(\mathbf{r} \mid q, m, p) = (1/\gamma)(\mathbf{S}^{-1}\mathbf{p} - \boldsymbol{\Sigma}^{-1}\boldsymbol{\pi})'\mathbf{r} - (1/2\gamma)\,[\,\mathbf{r}'(\mathbf{S}^{-1} - \boldsymbol{\Sigma}^{-1})\mathbf{r} + n - \mathrm{Tr}(\boldsymbol{\Sigma}\mathbf{S}^{-1})] \qquad \text{(A1)}$$

and

$$D(q, m, p) = \tfrac{1}{2}(\mathbf{p} - \boldsymbol{\pi})'\boldsymbol{\Sigma}^{-1}(\mathbf{p} - \boldsymbol{\pi}) + \tfrac{1}{2}\,[\mathbf{p}'(\mathbf{S}^{-1} - \boldsymbol{\Sigma}^{-1})\mathbf{p} + \boldsymbol{\pi}'(\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1})\boldsymbol{\pi}] + \tfrac{1}{2}\,[\,\mathrm{Tr}(\boldsymbol{\Sigma}\mathbf{S}^{-1}) - n - \ln(|\boldsymbol{\Sigma}\mathbf{S}^{-1}|)] \qquad \text{(A2)}$$

where $\mathrm{Tr}(.)$ denotes a trace and $|.|$ a determinant

Special MVN Case with S = Σ

When $\mathbf{S} = \boldsymbol{\Sigma}$, the terms in square brackets in (A1) and (A2) are nil, therefore the optimal net active portfolio is

$$f(\mathbf{r}\,|\,q, m, p) = (1/\gamma)\,(\mathbf{p} - \boldsymbol{\pi})'\boldsymbol{\Sigma}^{-1}\mathbf{r} \qquad \text{(A3)}$$

And the corresponding gain in certainty equivalent

$$(1/\gamma)D(q, m, p) = \tfrac{1}{2}(\mathbf{p} - \boldsymbol{\pi})'(\gamma\boldsymbol{\Sigma}^{-1})(\mathbf{p} - \boldsymbol{\pi}) \qquad \text{(A4)}$$

The reference portfolio corresponding to $N(\pi, \Sigma)$ is itself

$$f(\mathbf{r}\,|\,q, m) = \boldsymbol{\pi}'(\gamma\boldsymbol{\Sigma})^{-1}\mathbf{r} = \boldsymbol{\omega}'\mathbf{r} \qquad \text{(A5)}$$

with $\boldsymbol{\omega} = (\gamma\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi}$, i.e., same result as with mean variance analysis,

and its CE is

$$(1/\gamma)D(q, m) = \tfrac{1}{2}\boldsymbol{\pi}'(\gamma\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} \qquad \text{(A6)}$$

LDD with Views on Expected Returns Only

With views on expected returns only (condition (15)) and $D(q, m, p)$ given by (A2), we find that the minimum distance is obtained when choosing $\mathbf{S} = \Sigma$. The Lagrangian becomes

$$L = \tfrac{1}{2}(\mathbf{p} - \boldsymbol{\pi})'\Sigma^{-1}(\mathbf{p} - \boldsymbol{\pi}) - \lambda_1'(\mathbf{P}\mathbf{p} - \mathbf{q}) \qquad \text{(A7)}$$

and the minimum is obtained when

$$\partial \mathbf{L}/\partial \mathbf{p} = \Sigma^{-1}(\mathbf{p} - \boldsymbol{\pi}) - \mathbf{P}'\lambda_1 = 0$$

hence, $\mathbf{p} - \boldsymbol{\pi} = \Sigma\mathbf{P}'\lambda_1$

and $\lambda_1 = (\mathbf{P}\Sigma\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi})$

so that

$$\mathbf{p} = \boldsymbol{\pi} + \Sigma\mathbf{P}'(\mathbf{P}\Sigma\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi}) \qquad \text{(A8)}$$

Precisely the same result as with BLC (9) so imposing BL's choice of uncertainty parameter $\tau\Sigma$ as a necessity

Matching Active Pay-off and CE

The optimal active pay off with views on expected returns only is:

$$f(\mathbf{r}\,|\,q, m, p) = (\mathbf{q} - \mathbf{P}\boldsymbol{\pi})'(\mathbf{P}\gamma\Sigma\mathbf{P}')^{-1}\mathbf{P}\mathbf{r} = \boldsymbol{\omega}'\mathbf{r} \qquad \text{(A9)}$$

It is the linear combination of risky assets with weights:

$$\boldsymbol{\omega} = \mathbf{P}'(\mathbf{P}\gamma\Sigma\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi}) \qquad \text{(A10)}$$

The corresponding gain in certainty equivalent is according to (A2):

$$(1/\gamma)D(q, m, p) = \tfrac{1}{2}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi})'(\mathbf{P}\gamma\Sigma\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\pi}) \qquad \text{(A11)}$$

LDD with Views on Covariances

When minimizing the Lagrangian

$$L = D(q, m, p) - \boldsymbol{\lambda}_1'(\mathbf{Pp} - \mathbf{q}) - \boldsymbol{\lambda}_2'(\mathbf{G}.\mathrm{Vech}(\mathbf{S}) - \mathbf{h})$$

with $D(q, m, p)$ given in (A2), we find that whatever optimum parameters are found for $\mathbf{S}$, they will affect the choice of expected returns $\mathbf{p}$. Indeed

$$\partial \mathbf{L} / \partial \mathbf{p} = \boldsymbol{\Sigma}^{-1}(\mathbf{p} - \boldsymbol{\pi}) + (\mathbf{S}^{-1} - \boldsymbol{\Sigma}^{-1})\mathbf{p} = 0$$

leads to the solution

$$\mathbf{p} = \mathbf{S}\boldsymbol{\Sigma}^{-1}\boldsymbol{\pi} \qquad \text{(A12)}$$

A view on covariances affects therefore the expected returns that are not specified in addition to affecting other covariances

The resulting optimal pay-offs are quadratic in $\mathbf{r}$